CALCULATION OF THE RADIATION OF AN ISOTHERMAL GAS – DUST MEDIUM

Yu. A. Popov

UDC 536.3

A method is proposed for calculating radiative heat transfer in homogeneous isothermal gas – dust media.

1. For simplicity we will consider the one-dimensional problem and neglect scattering.

The total emittance of a homogeneous isothermal gas–dust medium is written in the form

$$\varepsilon(L,\ T) = 1 - \frac{1}{\sigma T^4}\int_0^\infty B_\nu(T)\, e^{-(K'_\nu(T)+K''_\nu(T))L} d\nu. \tag{1}$$

We will consider the transmittance of the gas having temperature T for thermal radiation with temperature T_i to be known

$$D_g(L,\ T,\ T_i) = \frac{1}{\sigma T_i^4}\int_0^\infty B_\nu(T_i) e^{-K''_\nu(T)L} d\nu. \tag{2}$$

The value of D is related with absorptance A by the relationship

$$D(L,\ T,\ T_i) + A(L,\ T,\ T_i) = 1, \tag{3}$$

The absorptance of CO_2 and H_2O can be found from empirical formulas and graphs, which are given in [1].

We will represent the function $B_\nu(T)e^{-K'_\nu L}$ in the form of a superposition of Planck's function with different temperatures

$$B_\nu(T)\, e^{-K'_\nu L} \simeq \sum_{i=1}^{N} a_i B_\nu(T_i). \tag{4}$$

In the general case the coefficients a_i and T_i depend on L and T. The coefficient T_i can always be selected so that Eq. (4) is fulfilled exactly for certain frequencies ν_n ($\nu_1, \nu_2, \ldots, \nu_N$) if $\nu_n \neq \infty$. This statement follows from the fact that for fixed ν_n and T_i the system of linear equations in a_i, which is obtained from (4), can be solved only if the determinant of this system is nonzero. But T_i can be selected such that the determinant of the system obtained is not equal to zero. Since the absorption coefficient of dust is a continuous and sufficiently smooth function of frequency, and functions $B_\nu(T_i)$ are also continuous and sufficiently smooth functions of frequency, we can state that, upon an increase of the number of terms in the expansion of (4) the accuracy of the expansion will increase if the expansion is done by the method described above.

Using (4), we obtained from (1) for the emittance the expression

$$\varepsilon(L,\ T) = 1 - \frac{1}{T^4}\sum_{i=1}^{N} a_i T_i^4 D_g(L,\ T,\ T_i). \tag{5}$$

Polytechnic Institute, Chelyabinsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 3, pp. 512-519, September, 1969. Original article submitted October 4, 1968.

As an example let us consider the case where the absorption coefficient of dust is proportional to the radiation frequency

$$K'_\nu = \alpha\nu. \tag{6}$$

If Eq. (4) is valid exactly in the Rayleigh–Jeans region, then with consideration of (6) we find

$$\sum_{i=1}^{N} a_i T_i = T. \tag{7}$$

We will take only two terms in the expansion of (4). For (4) to be fulfilled in the Wien region, it is necessary that

$$a_1 = 1, \quad T_1 = \frac{T}{1+z}, \tag{8}$$

where

$$z = \frac{kT}{h}\alpha L, \quad T_2 < T_1. \tag{9}$$

We assume that

$$T_2 = \frac{T}{\beta}. \tag{10}$$

Then it follows from (7) that

$$a_2 = \frac{z\beta}{1+z}. \tag{11}$$

After this we obtain from (5)

$$\varepsilon(L, T) = 1 - \frac{1}{(1+z)^4} D_g(L, T, T_1) - \frac{z}{(1+z)\beta^3} D_g(L, T, T_2). \tag{12}$$

We determine the coefficient β from the condition that, for the emittance of a layer of particles with absorption coefficient (6), we obtain an exact result

$$\varepsilon_p = 1 - \frac{1}{\zeta(4)} \sum_{i=1}^{\infty} \frac{1}{(i+z)^4}, \tag{13}$$

where

$$\zeta(4) = \sum_{i=1}^{\infty} i^{-4}.$$

Table 1 gives the total transmittance of a layer of particles D_p with absorption coefficient (6) as a function of the parameter z.

For coefficient β we obtain the expression

$$\beta = (1+z)\left[\frac{z}{(1+z)^4 D_p(z) - 1}\right]^{1/3}. \tag{14}$$

If we assume that the transmittance of the gas does not depend on the temperature of the radiator T_i, from (5) follows the known formula

$$\varepsilon = 1 - D_p D_g(L, T, T). \tag{15}$$

TABLE 1. Transmittance of a Layer of Particles with Absorption Coefficient (6) as a Function of the Dimensionless Parameter z (9)

z	D_p	z	D_p	z	D_p	z	D_p
0	1	0,40	0,2803	0,8	0,1112	1,6	0,0301
0,05	0,8297	0,45	0,2460	0,9	0,0915	1,7	0,0264
0,10	0,6949	0,50	0,2170	1,0	0,0761	1,8	0,0232
0,15	0,5870	0,55	0,1922	1,1	0,0639	1,9	0,0206
0,20	0,4997	0,60	0,1710	1,2	0,0541	2,0	0,0183
0,25	0,4284	0,65	0,1528	1,3	0,0462	2,5	0,0108
0,30	0,3697	0,70	0,1370	1,4	0,0398	3,0	0,0069
0,35	0,3210	0,75	0,1233	1,5	0,0345	4,0	0,0033

TABLE 2. Emittance of Dust-Laden Water Vapor with an Absorption Coefficient of the Particles $K' = \alpha\nu$ as a Function of the Dimensionless Parameter z (9) (L = 0.9 m, P = 1 atm, T = 1800°K)

z	ε	$1-D_p D_g(L, T, T)$	z	ε	$1-D_p D_g(L, T, T)$
0	0,22	0,22	1,0	0,955	0,941
0,1	0,48	0,46	2,0	0,990	0,986
0,5	0,85	0,83	3,0	0,9968	0,9947

Table 2 gives the results of calculating the emittance of a layer of water vapor 0.9 m thick at a pressure of 1 atm and temperature of 1800°K containing dust particles with absorption coefficient (6). The calculation was made by Eqs. (12) and (15). The results for emittance agree well with one another. But we see from Table 2 that the accuracy of calculating the transmittance of the gas–dust medium by the approximate formula

$$D_{pg} \simeq D_p D_g(L,\ T,\ T) \tag{16}$$

decreases rapidly with increase of the concentration of particles, which is proportional to the parameter z.

Let us consider another example. Let the absorption coefficient of the particles have a form such that for fixed L and T the equation

$$\frac{\nu^3 e^{-K_\nu L}}{e^{\frac{h\nu}{kT}} - 1} = a_1 \frac{\nu^3}{e^{\frac{h\nu}{kT_1}} - 1}. \tag{17}$$

holds true. Let $K'_\nu \to 0$ when $\nu \to 0$, then

$$a_1 = \frac{T}{T_1}. \tag{18}$$

Introducing the notation

$$m = \frac{T}{T_1}, \tag{19}$$

we can write

$$e^{-K'_\nu L} = m\,\frac{e^{\frac{h\nu}{kT}} - 1}{e^{m\frac{h\nu}{kT}} - 1}. \tag{20}$$

The total transmittance of the layer of particles whose absorption coefficient satisfies Eq. (20) has the form

$$D_p = \frac{1}{m^3}, \tag{21}$$

and the transmittance of the gas–dust medium for thermal radiation having the temperature of the medium is written with consideration of (20), (21), and (5) in the form

$$D_{gp} = \frac{1}{m^3} D_g\left(L,\ T,\ \frac{T}{m}\right). \tag{22}$$

TABLE 3. Transmittance of Dust-Laden Water Vapor with an Absorption Coefficient of the Particles Satisfying Eq. (20) as a Function of the Dimensionless Parameter m (L = 3 m, P = 1 atm, $T = T_0 = 1200°K$)

m	D_{gp}	$D_p\,D_g\,(L,\ T,\ T)$
1,0	0,56	0,56
1,2	0,29	0,33
1,5	0,125	0,167
2,0	0,048	0,070

TABLE 4. Reflectance of Semiinfinite Gas–Dust Medium ($T = T_0 = 1000°K$, P = 1 atm, $K_s = K'$)

K_s, cm^{-1}	R_∞ (CO_2)	R_∞ (H_2O)
∞	0,170	0,170
1,0	0,163	0,167
0,1	0,154	0,154
0,05	0,150	0,137
0,01	0,141	0,119

Table 3 presents the results for the transmittance of a gas–dust medium calculated by formulas (22) and (16). The results are given for water vapor at T = 1200°K, P = 1 atm, and L = 3 m laden with particles whose absorption coefficient satisfies Eq. (20). We can also conclude from Table 3 that calculation of the transmittance of the gas–dust medium by Eq. (16) for the case where D_{pg} is much less than 1 for large pL of the absorbing gas can lead to appreciable errors.

Obviously the method presented can be used for calculating the transmittance of homogeneous isothermal gas–dust media also in the case where the temperature of the incident radiation differs from the temperature of the medium

$$D_{gp}(L,\ T,\ T_0) = \frac{1}{\sigma T_0^4}\int\limits_0^\infty B_\nu(T_0)\, e^{-(K_\nu''(T)+K_\nu'(T))L}\, d\nu. \tag{23}$$

For this purpose it is necessary to present $B_\nu(T_0)e^{-K_\nu' L}$ in the form of a superposition of Planck's functions with different temperatures.

2. If scattering of radiation by the particles of the medium cannot be neglected, the problem of radiative heat transfer in a homogeneous gas–dust medium is conveniently solved by means of the distribution density function of photons over the paths traversed by them in the medium. The distribution density function of photons over the paths was first studied in the works of Van de Hulst and Irvine [2, 3].

For simplicity we will consider the one-dimensional problem of radiative transfer through a layer [4]. Knowing the solution of the one-dimensional problem, it is easy to obtain the solution for a plane layer in a Schwarzschild–Schuster approximation. We will consider that the layer is bounded by black walls, and radiation of unit intensity falls on the surface $\tau = 0$. The probability that a photon with frequency ν will be reflected from the medium, traversing in the medium a path of optical thickness from τ_ν to $\tau_\nu + d\tau_\nu$, will be written in the form

$$dW_{1\nu} = f_1(\lambda_\nu,\ \tau_{0\nu},\ \tau_\nu)\, d\tau_\nu,$$

where f_1 is the distribution density function over the paths for reflected photons; λ_ν is the ratio of the scattering coefficient to the attenutation coefficient; $\tau_{0\nu}$ is the spectral optical thickness of the layer. Likewise for photons passing through the layer we write

$$dW_{2\nu} = f_2(\lambda_\nu,\ \tau_{0\nu},\ \tau_\nu)\, d\tau_\nu.$$

For the intensity of the reflected and transmitted radiation we will have respectively

$$I_{1\nu} = \int\limits_0^\infty f_1(\lambda_\nu,\ \tau_{0\nu},\ \tau_\nu)\, d\tau_\nu;\quad I_{2\nu} = \int\limits_0^\infty f_2(\lambda_\nu,\ \tau_{0\nu},\ \tau_\nu)\, d\tau_\nu. \tag{24}$$

We will now establish the relation between the distribution density function over the paths in the case of pure scattering $\lambda_\nu = 1$ and the distribution density function over the paths for $\lambda_\nu \neq 1$. For both cases we will consider the same photon trajectory in real space. For $\lambda_\nu \neq 1$ the probability that the photon will traverse path τ_ν to emergence from the medium will be less than the analogous quantity for pure scattering by $\exp[-(1-\lambda_\nu)\tau_\nu]$ times. Taking into account that the optical path in a nonabsorbing medium is λ_ν times less than the optical path in the same medium with absorption, we obtain

$$f(\lambda_\nu,\ \tau_{0\nu},\ \tau_\nu) = \lambda_\nu e^{-(1-\lambda_\nu)\tau_\nu} f(1,\ \lambda_\nu\tau_{0\nu},\ \lambda_\nu\tau_\nu). \tag{25}$$

As follows from the derivation, Eq. (25) is general and is valid also for a three-dimensional medium. Using (25) and (24), for the intensity and the boundaries of the medium we obtain the known expression [2, 3, 5]

$$I_\nu(s,\ x_0) = \int_0^\infty e^{-sx} f(1,\ x_0,\ x)\, dx, \tag{26}$$

where the notations

$$x = \lambda_\nu \tau_\nu;\quad x_0 = \lambda_\nu \tau_{0\nu};\quad s = \frac{1-\lambda_\nu}{\lambda_\nu}. \tag{27}$$

are introduced.

Thus, knowing the intensity at the boundaries of the medium, we can find the distribution density function of photons over the paths in the case of pure scattering by resorting to Laplacian transformation.

Let the boundary of the medium $\tau = 0$ be illuminated by thermal radiation with temperature T. According to (24) and (25), for the intensity of the reflected and transmitted radiation we will have

$$I_{1,2} = \int_0^\infty B_\nu(T_0) \left[\int_0^\infty e^{-(K'_\nu + K''_\nu(T))l} f_{1,2}(K_{s\nu}, L, l)\, dl\right] d\nu. \tag{27'}$$

In deriving (27') we did not introduce the optical thickness and we denote by l the path of the photon in the medium up to its emergence from the medium. In this case, if the absorption coefficient of dust K' and the scattering coefficient K_S do not depend on the radiation frequency ν, we obtain

$$I_{1,2} = \int_0^\infty \varphi(l,\ T,\ T_0)\, e^{-K'l} f_{1,2}(K_s,\ L,\ l)\, dl. \tag{28}$$

If the function

$$\varphi(l,\ T,\ T_0) = \sigma T_0^4 D_g(l,\ T,\ T_0) \tag{29}$$

can be represented in the form of a set of exponents

$$D_g(l,\ T,\ T_0) = \sum_{i=1}^{n} b_i e^{-K_i l}, \tag{30}$$

we find from (28)

$$I_{1,2} = \sigma T_0^4 \sum_{i=1}^{n} b_i I_{0i\,1,2}, \tag{31}$$

where

$$I_{0i\,1,2} = \int_0^\infty e^{-(K'+K_i)l} f_{1,2}(K_s,\ L,\ l)\, dl. \tag{32}$$

According to (24) and (25), $I_{0i\,1,2}$ is the radiation intensity at the boundary of the medium for sulfur–gas medium with absorption coefficient $K' + K_i$ and scattering coefficient K_S. The result, described by Eq. (31), is valid also in the case of a three-dimensional medium. It is analogous to Hottel's result [1] for radiative heat transfer in a gas medium bounded by walls whose reflectance does not depend on the radiation frequency.

If we cannot neglect the frequency dependence of K'_ν and $K_{\nu S}$, in the calculation of (27) we can use the method presented in Section 1, i.e., represent the function

$$B_\nu(T_0)\, e^{-K'_\nu l} f(K_{s\nu},\ L,\ l)$$

in the form of superposition of Planck's functions with different temperatures.

3. As an example, we will consider the problem of reflection of thermal radiation from an isothermal homogeneous gas-dust medium in a Schwarzschild–Schuster approximation. We will consider the scattering indicatrix to be spherical. If radiation of unit intensity is incident upon the medium, the intensity of the reflected radiation has the form (e.g., [6])

$$I = \frac{1-\sqrt{1-\lambda}}{1+\sqrt{1+\lambda}}. \tag{33}$$

Using (27) and generalizing the Laplacian transformation (26), we can obtain by the contour integration method

$$f(1,\ \tau) = \frac{2}{\pi} \int_0^1 e^{-(1-y)\tau} \sqrt{y(1-y)}\, dy. \tag{34}$$

Equation (34) was first obtained by Sobolev in [4] in connection with the problem of nonsteady-state diffusion in gas. For simplification of Eq. (34) we will use the following representation of Bessel's function of the imaginary argument I_1:

$$I_1(z) = \frac{4}{\pi} z e^z \int_0^1 e^{-2(1-y)z} \sqrt{y(1-y)}\, dy, \tag{35}$$

which is easily obtained from the known integral representation. With consideration of Eq. (35) instead of (34) we obtain

$$f(1,\ \tau) = \frac{e^{-\tau/2}}{\tau} I_1(\tau/2). \tag{36}$$

Using (28) and (29), we write for the reflectance of a semiinfinite gas–dust medium the expression

$$R_\infty = K_s \int_0^\infty D_g(l,\ T,\ T_0)\, e^{-K'l} \frac{e^{-\frac{K_s l}{2}}}{K_s l} I_1\left(\frac{K_s l}{2}\right) dl. \tag{37}$$

We assume that $K_S \neq 0$, and introduce the new variable $K_S l = \eta$:

$$R_\infty = \int_0^\infty D_g\left(\frac{\eta}{K_s},\ T,\ T_0\right) e^{-\frac{K'}{K_s}\eta} \frac{e^{-\eta/2}}{\eta} I_1\left(\frac{\eta}{2}\right) d\eta. \tag{38}$$

The integral is easily calculated by the method of quadratures. The table of Bessel's function of an imaginary argument is given in [7].

Table 4 presents a calculation of the reflectance of a semiinfinite layer of dust-laden carbon dioxide at 1000°K and pressure P = 1 atm, and also analogous data for dust-laden water vapor for the same parameters. The calculation was performed for the case $K' = K_S$ for some values of K_S for thermal radiation having the temperature of the medium T = 1000°K. In making these calculations we used the quadrature formula with a weight function e^{-x} with three points [8]. The sufficient accuracy of the quadrature formula used is confirmed by the fact that when $K_S \to \infty$ or, what is equivalent, when $D_g = 1$, the result practically coincides with the calculation by Eq. (33) for $\lambda = 0.5$.

NOTATION

- ε is the emittance;
- D_g is the transmittance of gas;
- A is the absorptance;
- ε_p is the emittance of cloud of particles;
- T is the temperature of gas–dust medium;
- ν is the radiation frequency;
- B_ν is the Planck's constant;
- K'_ν is the absorption coefficient of particles;
- K''_ν is the absorption coefficient of gas;
- f is the distribution density function of photons over paths;
- λ is the ratio of scattering coefficient to attenuation coefficient;
- $\tau_{0\nu}$ is the optical thickness of layer;
- τ_ν is the optical path traversed by a photon in the medium to emergence from the medium;
- I is the radiation intensity;
- L is the layer thickness;
- l is the path length traversed by a photon in the medium before emerging from the medium;
- T_0 is the radiation temperature;
- D_{gp} is the transmittance of gas–dust medium;

h	is the Planck constant;
k	is the Boltzmann constant;
σ	is the Stefan–Boltzmann constant;
D_p	is the transmittance of dust medium;
R_∞	is the reflectance of semiinfinite gas–dust medium;
I_1	is the first-order Bessel function of an imaginary argument;
P	is the gas pressure.

LITERATURE CITED

1. W. H. McAdams, Heat Transfer [Russian translation], Metallurgizdat (1961).
2. H. C. Van de Hulst and W. M. Irvine, Met. Soc. Roy. Sci. Liege, Ser. 5, 7, No. 1 (1963).
3. W. M. Irvine, Bull. Astron. Inst. Netherlandes, 17, 4 (1964); Astrophys. J., 144, 1140 (1966).
4. V. V. Sobolev, Radiative Transfer in Stellar and Planetary Atmospheres [in Russian], GITTL (1956)
5. L. M. Romanova, Izv. Akad. Nauk SSSR, Fiz. Atm. i Okeana, 1, No. 6, 599 (1965).
6. V. N. Adrianov, Teploénergetika, No. 2 (1961).
7. E. Janke, F. Emde, and F. Loesch, Special Functions [Russian translation], Nauka (1964).
8. V. I. Krylov and L. T. Shul'gina, Handbook on Numerical Integration [in Russian], Nauka (1966), p. 251.